Fractals, Scaling and Renormalization (1)

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Introduction

"Unfortunately, the world has not been designed for the convenience of mathematicians."

Benoît Mandelbrot

Born: 20 November 1924, Warsaw, Poland

Died: 14 October 2010 (aged 85), Cambridge, Massachusetts, United States

Image: At a TED conference in 2010.*@wikipedia*

Introduction

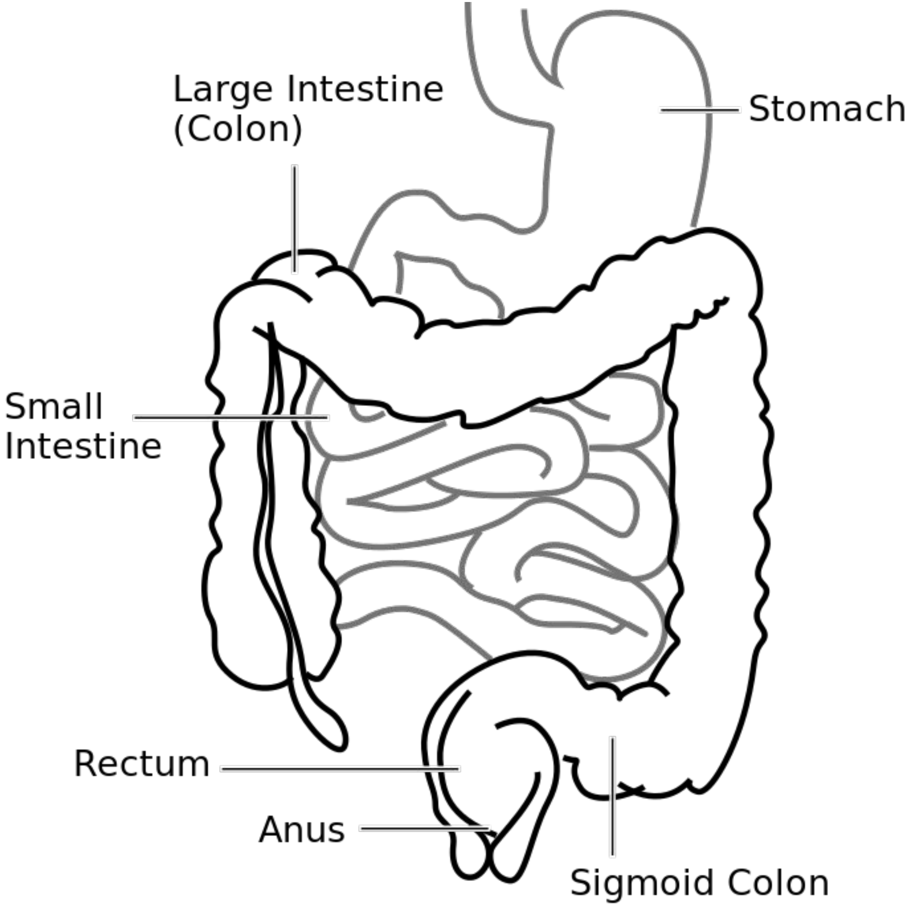
———

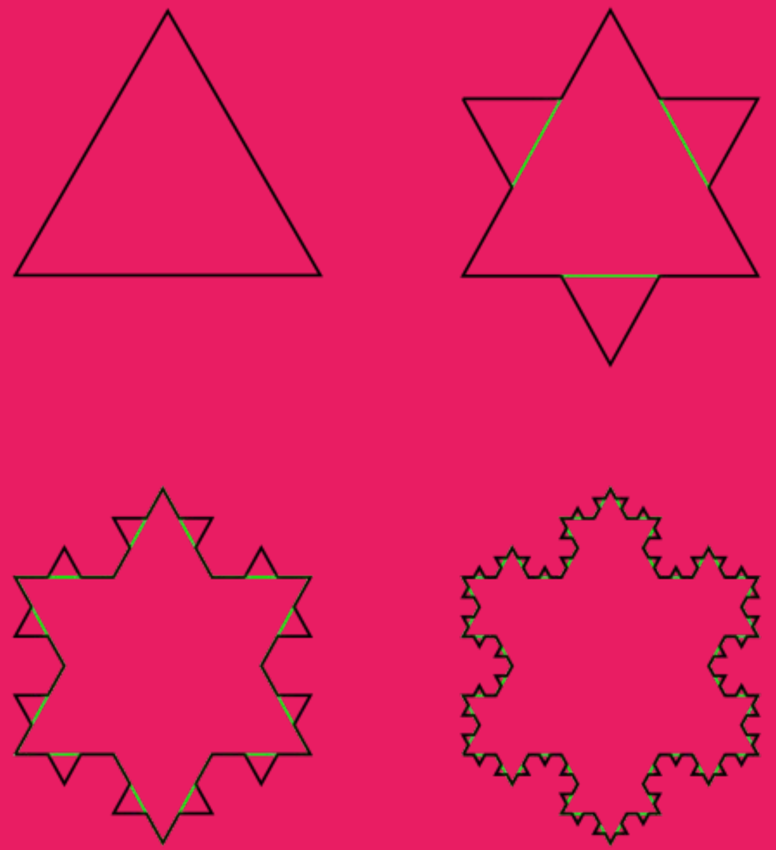
length of the small intestine:

2.75 – 10.49 m!

But, **HOW?**

Wikipedia





Wikipedia

$$s_1 = \frac{\sqrt{3}}{4}$$

$$s_2 = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{1}{3}\right)^n$$

$$s_3 = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 + \frac{12\sqrt{3}}{4}\left(\frac{1}{9}\right)^2$$

$$s_4 = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 + \frac{12\sqrt{3}}{4}\left(\frac{1}{9}\right)^2 + \frac{48\sqrt{3}}{4}\left(\frac{1}{27}\right)^2$$

.

.

.

$$S = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{9}\right)^2 + 4^2\left(\frac{1}{27}\right)^2 + \cdots\right) = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{4^0}{9^1} + \frac{4^1}{9^2} + \frac{4^2}{9^3} + \cdots\right)$$

$$so,\ S_n = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\sum_{i=2}^{n} \frac{4^{i-2}}{9^{i-1}}\right) = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\sum_{i=2}^{n} \left(\frac{4}{9}\right)^{i-1}\right)$$

$$S_\infty = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\frac{\frac{4}{9}}{\frac{5}{9}}\right) = \frac{2}{5}\sqrt{3}$$

$$p: 3 * 1,\ 3 * \left(4 * \frac{1}{3}\right),\ 3 * \left(16 * \frac{1}{9}\right),\ \ldots p_n = 3\left(\frac{4}{3}\right)^n$$

$$so\ if:\quad n \to \infty\ \ then\ p_n \to \infty$$

Self-Similarity

If we walk the coastline then we discover even **more fine detail.** Structures which exhibit a similar pattern whatever the scale are said to be **self-similar**.

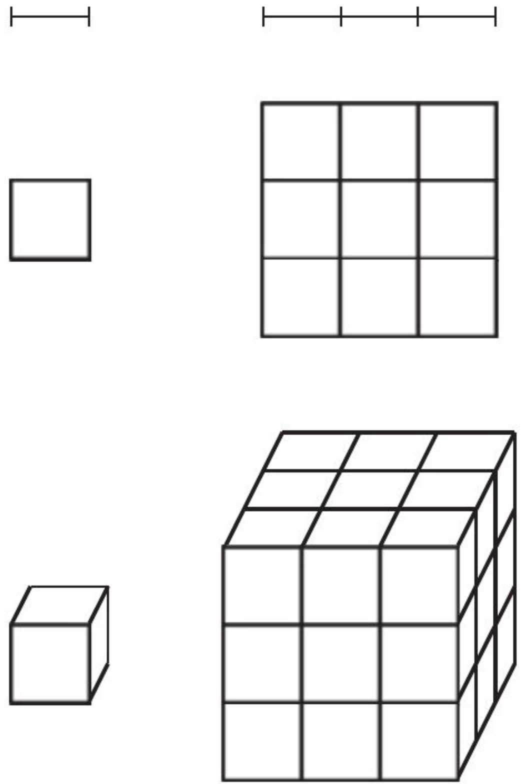
Self-Similarity

Wikipedia

Fractional Dimension!

How many little things fit inside a big thing?

Shape	M. Factor	Number of small copies
line	3	3
square	3	9
cube	3	27

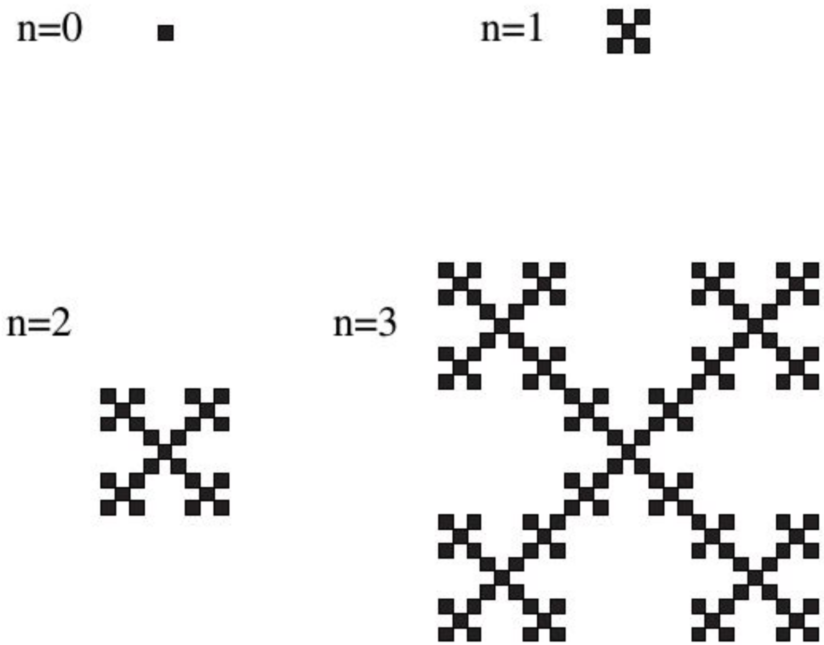


Snowflake

D = 1.46

Seriously?!

Why not D = 2 ?!



$$5=3^D \Longrightarrow D=\frac{\log 5}{\log 3}=1.46497$$

$$m^D=n \Rightarrow D=\frac{\log(n)}{\log(m)}=\log_m n$$

Wikipedia

D(Sierpiński Triangle) > D(Snowflake)

1.58 > 1.46

$$3=2^D \implies D=\frac{\log 3}{\log 2}=1.585$$

Wikipedia

Self-Similar Dimension, let's call it ROUGHNESS!

https://www.ted.com/talks/benoit_mandelbrot_fractals_the_art_of_roughness

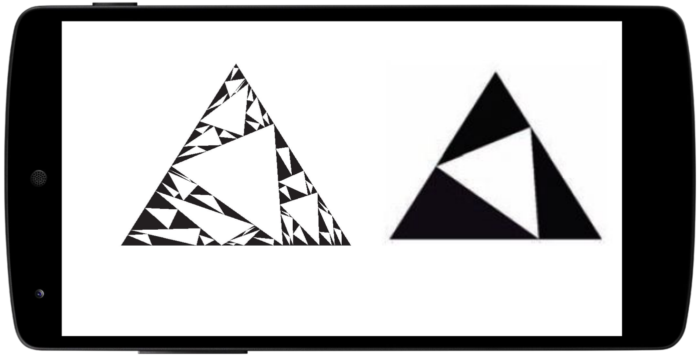
Self-Similarity VS Topological Dimension!

A fractal is a geometrical object whose self-similarity dimension is greater than its topological dimension.

- The Sierpinski triangle has a self-similarity dimension of **1.585** and a topological dimension of **1**.
- The Cantor set has a self-similarity dimension of **0.6309** and a topological dimension of **0**!

Random Fractals

1. Random Koch Curve
2. Chaos Game
3. Collage Theorem



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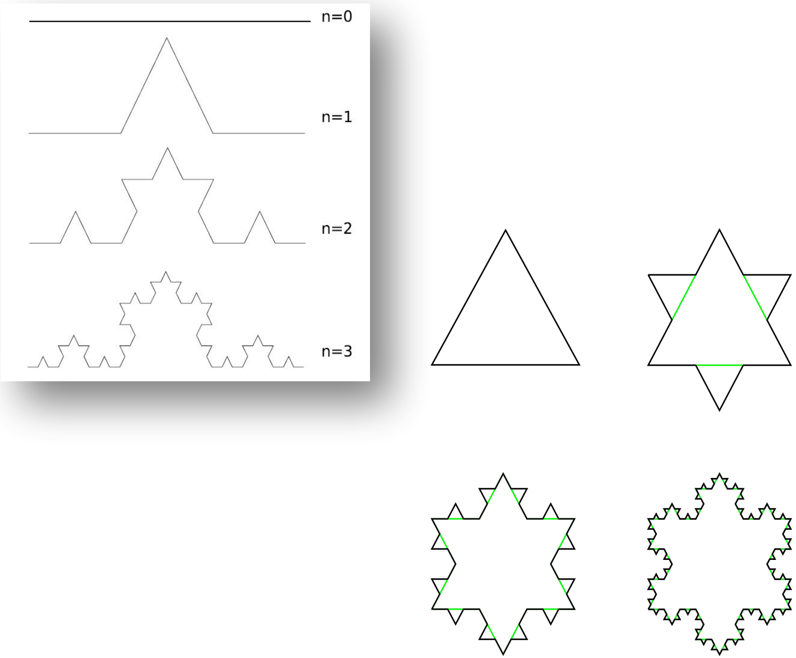
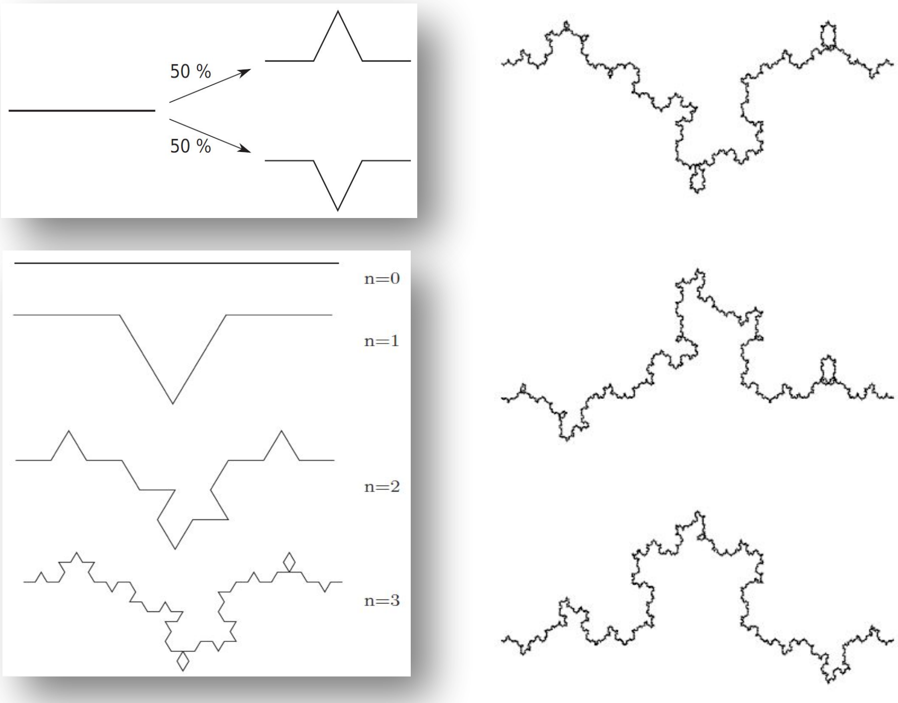




Koch Curve

A self-similar shape. A small copy of the curve, when magnified, **exactly** resembles the full curve. D=log4/log3.
See *Falconer, 2003, Chapter 15*

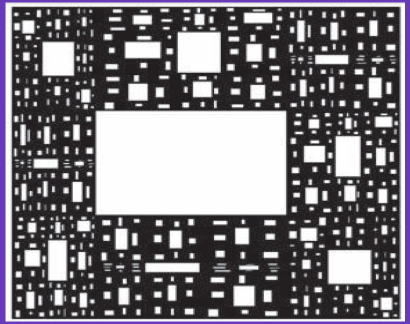
Random Koch Curve

A small copy of the curve, when magnified, closely resembles the full curve, but it is not an exact replica. (D=log4/log3)
Statistical self-similarity!

Irregular Fractals are not Random!

- *We make them by using an asymmetric or irregular generation rule.*
- *They are produced with a deterministic rule so they are NOT random!*
- *Nevertheless, the resultant shape has a somewhat random or disordered feel to it!*

Sierpiński Carpet is not random!



Minkowski–Bouligand (box-counting) Dimension!

How many little things fit inside a big thing?

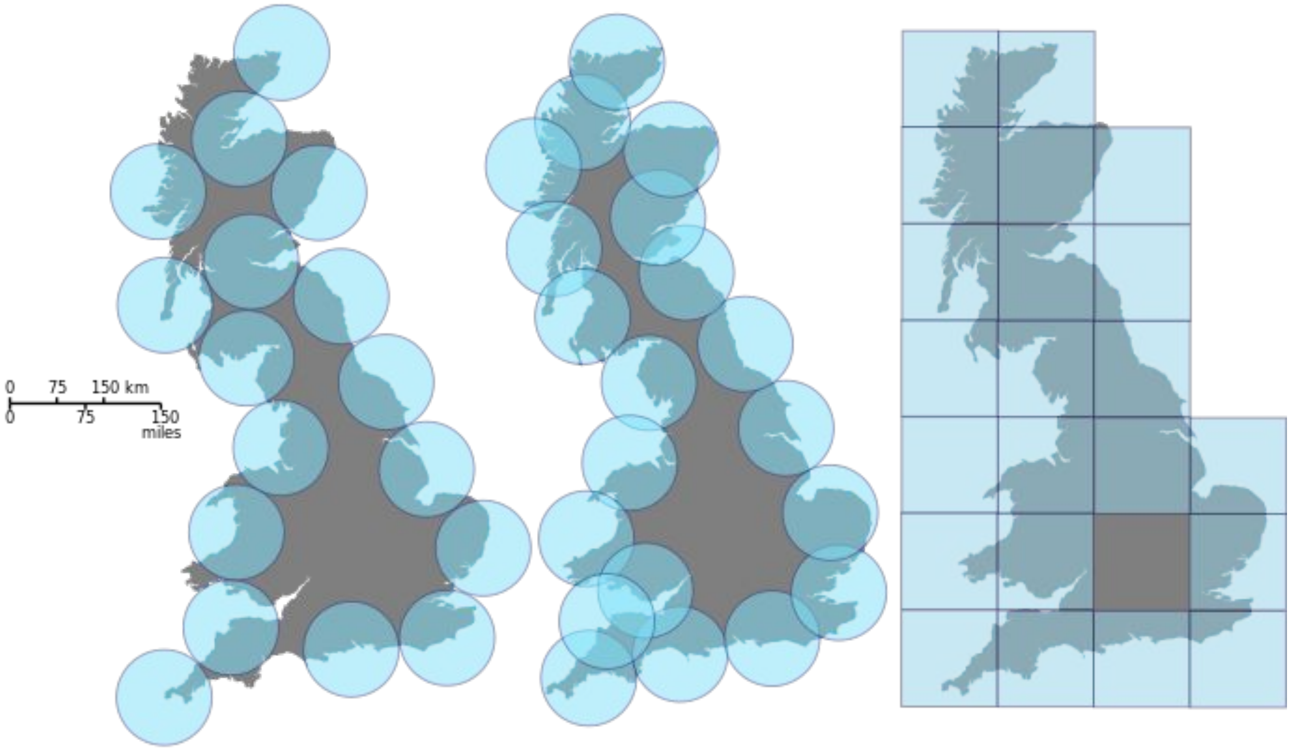
N(ε): number of boxes of side length ε

$$\dim_{\text{box}}(S) := \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}.$$

Wikipedia

Minkowski–Bouligand Dimension

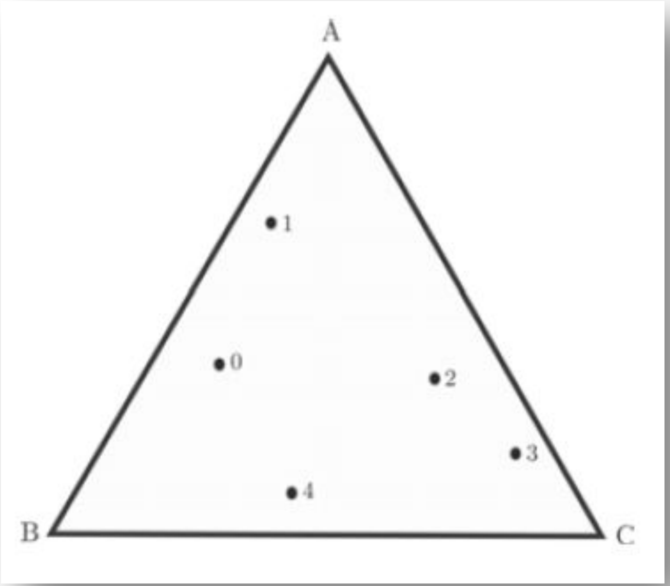
Examples of ball packing, ball covering, and box covering.



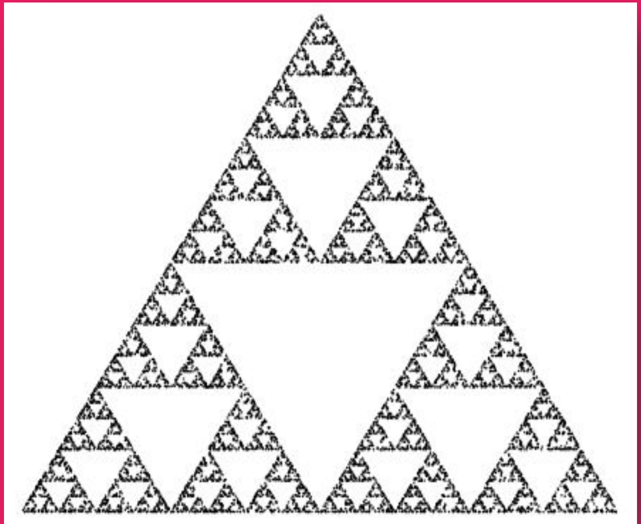
Wikipedia

Chaos Game

This is a stochastic (not deterministic) dynamical system; an element of chance is incorporated in each step!

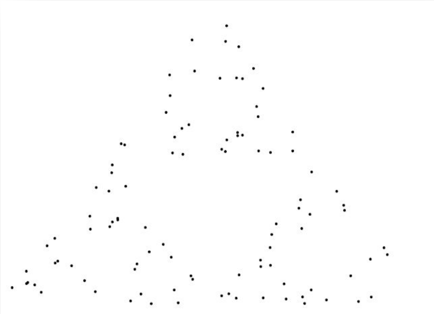


100,000 points

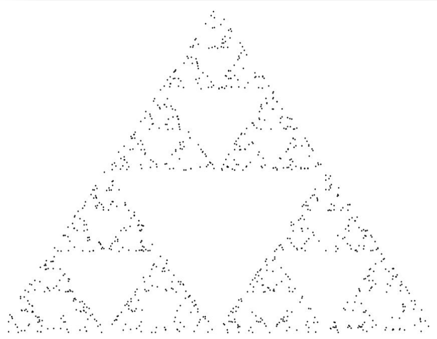


- ➔ *The chaos game shows that a random dynamical system can give a deterministic result.*
- ➔ ***#emergence_of_order***

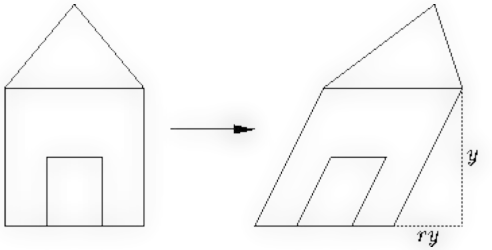
100 points



1000 points



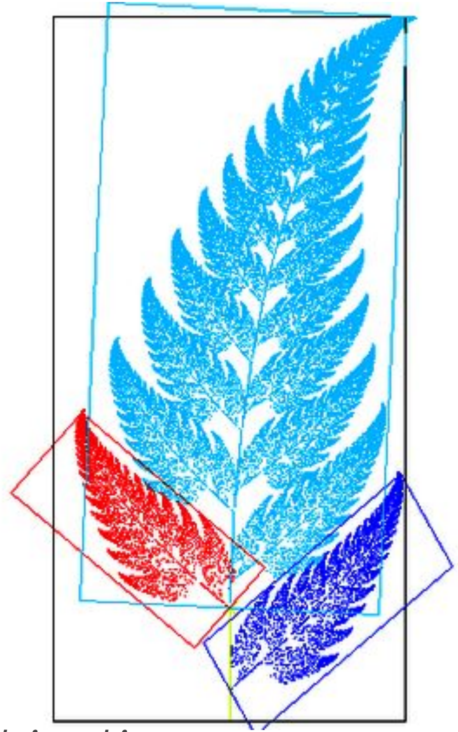
Collage Theorem



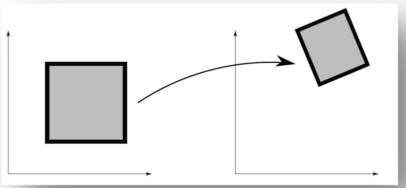
Given essentially any shape (not only fractals), one can make a chaos game that will generate it.

- Since in finding the particular affine transformations needed to reproduce an image, one forms a collage in which the full shape is covered with several smaller shapes. These smaller shapes then define the affine transformations that will generate the full shape.
- Application: Image compression

Collage Theorem

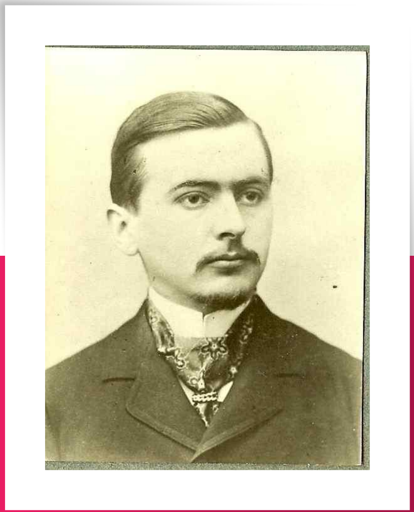


Wikipedia



An image of a fern-like fractal that exhibits affine self-similarity. Each of the leaves of the fern is related to each other leaf by an affine transformation.

The red leaf can be transformed into both the small dark blue leaf and the large light blue leaf by a combination of reflection, rotation, scaling, and translation.



Julia Set
Mandelbrot Set

Julia Set

The julia set for a function is simply the collection of all initial conditions that do not tend to infinity when iterated with that function.

- ❖ *2,4,16, …* diverging
- ❖ *-3, 9, 81, …* diverging

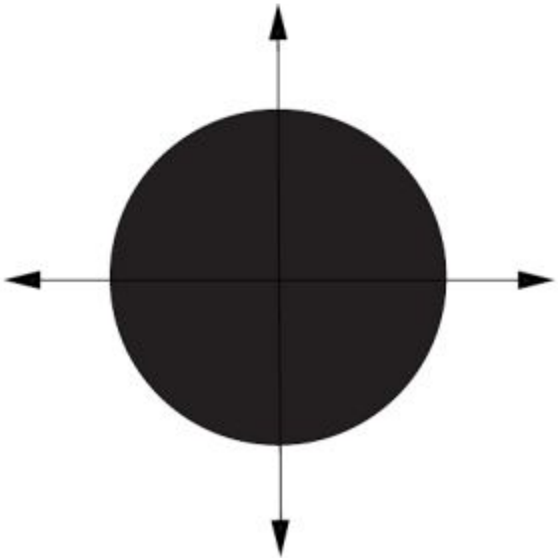
- ❖ *0.5, 0.25, 0.0625, … ,0.0* converging
  - ❖ Filled Julia Set = {-1,1}

$$y = x^2$$

$$x \in [-\infty, \infty]$$
$$x \in [-1, 1]$$

Julia Set, The complex squaring function

$$f(z) = z^2$$



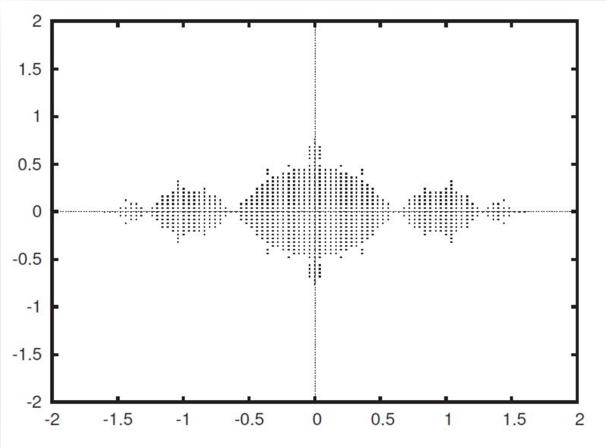
- Filled Julia Set is a **circle** with **r = 1**

Julia Set, What about this function?

$$f(z) = z^2 - 1$$

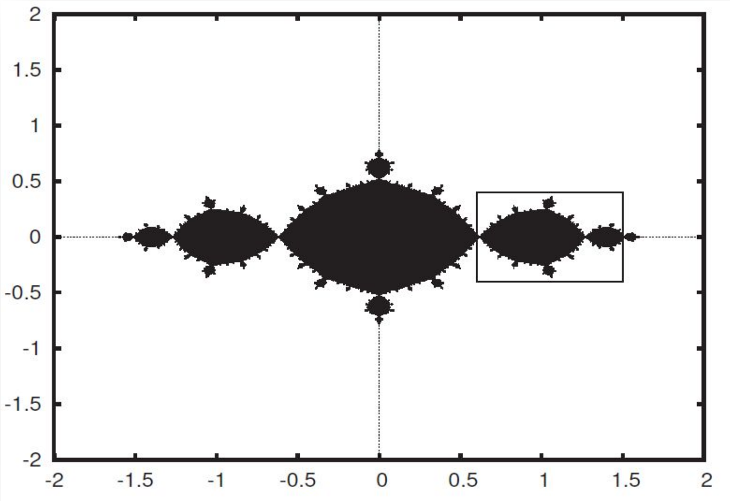
$$0 \longrightarrow -1 \longrightarrow 0 \longrightarrow -1 \longrightarrow \cdots .$$

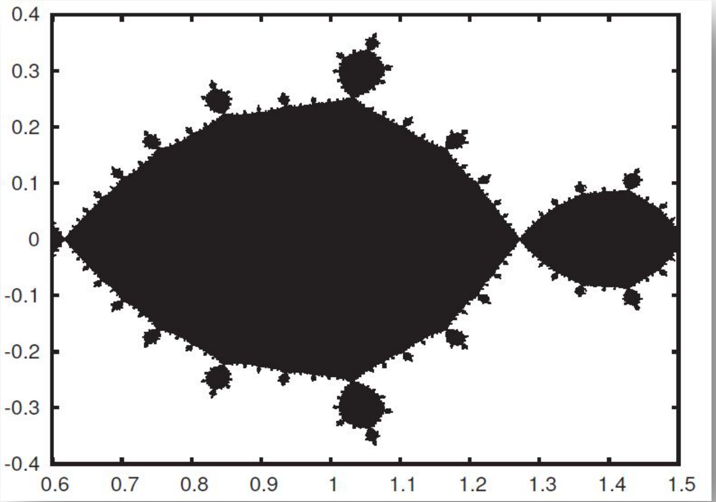
$$i \longrightarrow -2 \longrightarrow 3 \longrightarrow 8 \longrightarrow 63 \longrightarrow \cdots .$$



Julia Set, What about this function?

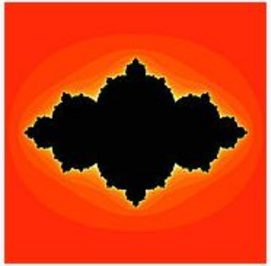
$$f(z) = z^2 - 1$$



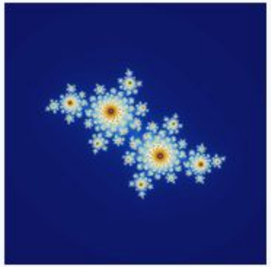


Julian Sets for :

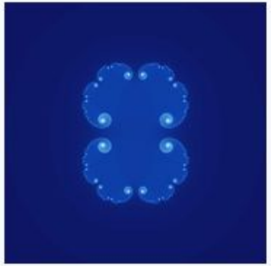
$$f(z) = z^2 + c$$



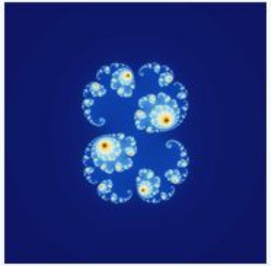
Filled Julia set for f_c, c=1−φ where φ is the golden ratio



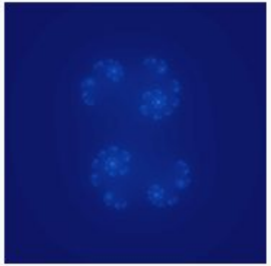
Julia set for f_c, c=(φ−2)+(φ−1)i =-0.4+0.6i



Julia set for f_c, c=0.285+0i

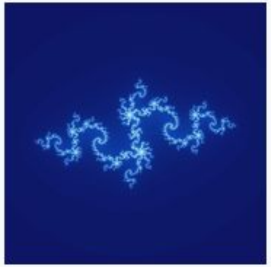


Julia set for f_c, c=0.285+0.01i

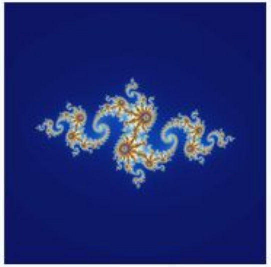


Julia set for f_c, c=0.45+0.1428i

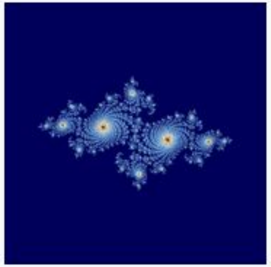
Julia set for f_c, c=-0.70176-0.3842i



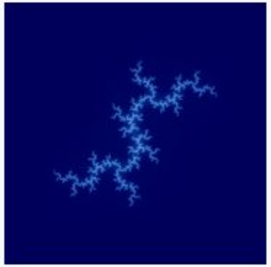
Julia set for f_c, c=-0.835-0.2321i



Julia set for f_c, c=-0.8+0.156i

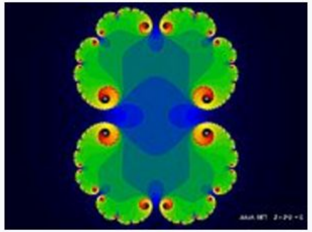


Julia set for f_c, c=-0.7269+0.1889i

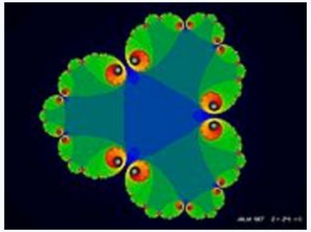


Julia set for f_c, c=-0.8i

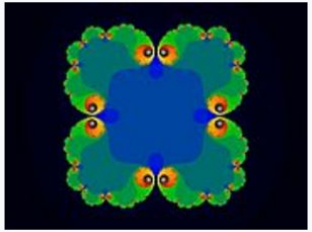
Other functions?



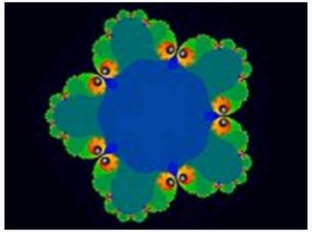
$f(z) = z^2 + 0.279$



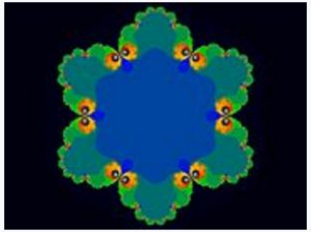
$f(z) = z^3 + 0.400$



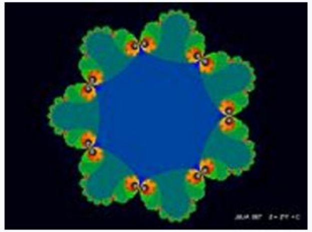
$f(z) = z^4 + 0.484$



$f(z) = z^5 + 0.544$



$f(z) = z^6 + 0.590$



$f(z) = z^7 + 0.626$

Wikipedia

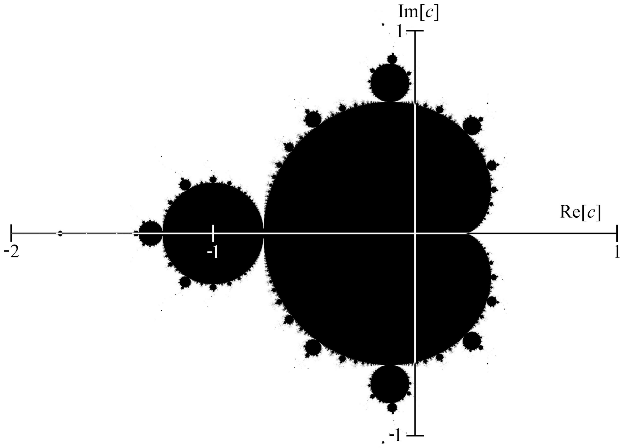
Mandelbrot Set

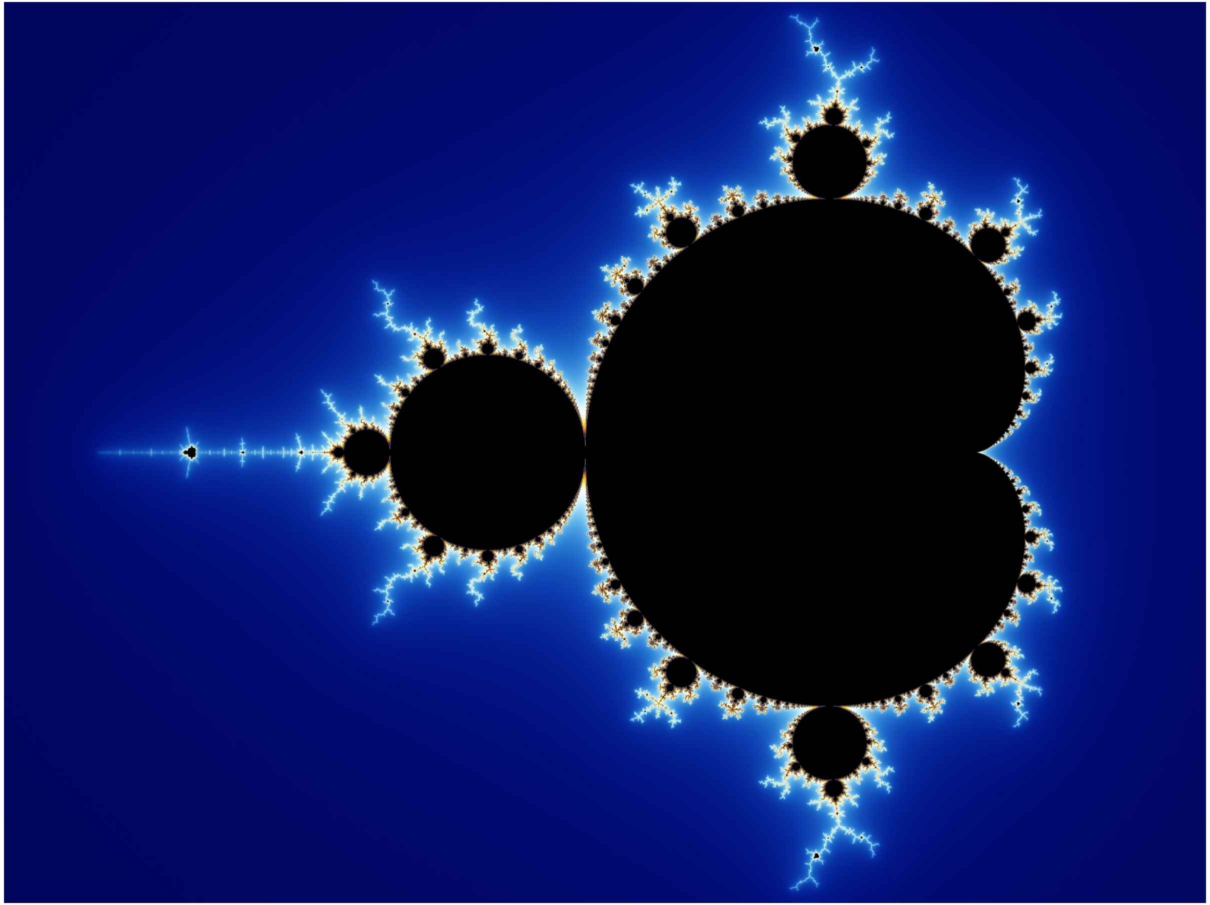
The Mandelbrot set consist of the set of all parameter values c for which the julia set of “ $f(z) = z^2 + c$ ” is connected.

➔ **Set of complex numbers *c* for which the function above does not diverge when iterated from z=0, i.e., for which the sequence remains bounded in absolute value:**

c, c^2+c, $(c^2+c)^2+c$, $((c^2+c)^2+c)^2+c$, $(((c^2+c)^2+c)^2+c)^2+c$, …

Mandelbrot Set





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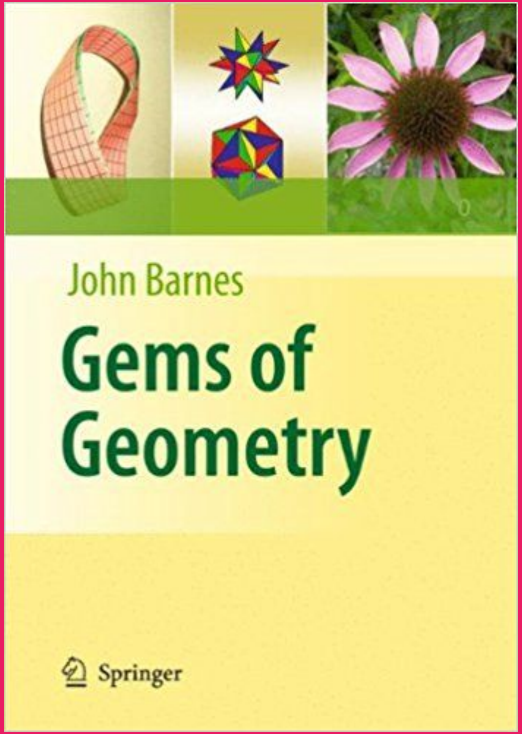

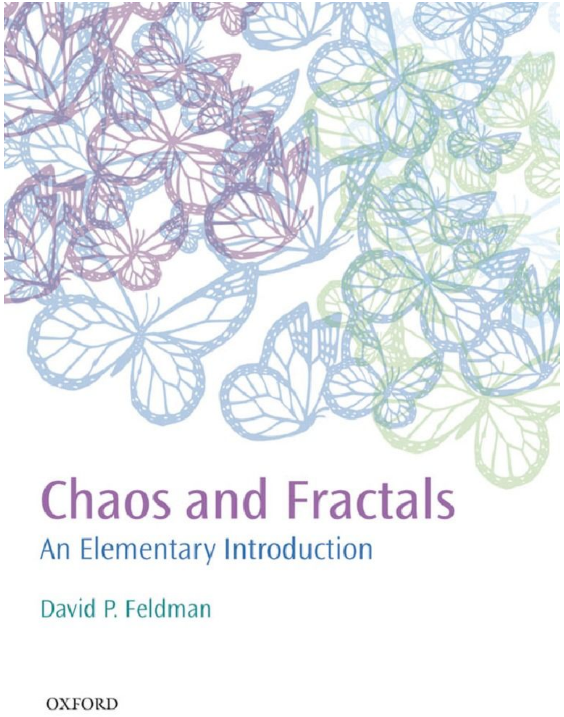

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Thank You :D

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October, 2018

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